Gauge-invariant gravitational perturbations of maximally symmetric spacetimes

Shinji Mukohyama
Department of Physics and Astronomy, University of Victoria
Victoria, BC, Canada V8W 3P6
(February 1, 2008)

Gravitational perturbations of anti-deSitter spacetime play important roles in AdS/CFT correspondence and the brane world scenario. In this paper, we develop a gauge-invariant formalism of gravitational perturbations of maximally symmetric spacetimes including anti-deSitter spacetime. Existence of scalar-type master variables is shown and the corresponding master equations are derived.

PACS numbers: 04.50.+h; 98.80.Cq; 12.10.-g; 11.25.Mj

I. INTRODUCTION

Recently, anti-deSitter (AdS) spacetime has been attracting a great deal of physical interests. In AdS/CFT correspondence, gravitational theory in AdS background is dual to a conformal field theory (CFT) on the boundary of the AdS [1,2]. It is believed that a correlation function in the CFT can be calculated by a path integral in the gravitational theory in AdS background with a certain boundary condition at the boundary. Moreover, the large N limit of the CFT is corresponding to the classical limit of the gravitational theory in AdS, where N is the number of colors. Therefore, the classical scattering of gravitational fields in AdS background is an important issue. In other words, it is important to investigate classical perturbations of AdS spacetime.

Another subject in which AdS spacetime plays important roles is the brane-world scenario. Randall and Sundrum [3] showed that, in a 5-dimensional AdS background, 4-dimensional Newton's law of gravity can be reproduced on a 4-dimensional timelike hypersurface despite the existence of the infinite fifth dimension. To be precise, they considered a 4-dimensional timelike thin-shell with its tension fine-tuned and showed that zero modes of gravitational perturbations are confined along the thin-shell and are decoupled from all non-zero modes in low energy. Therefore, it may be possible to consider the thin-shell, or the world volume of a 3-brane, as our universe, provided that matter fields can be confined on the 3-brane. In this respect, many authors investigated validity of the brane-world scenario from various points of view. For example, 4-dimensional effective Einstein equation on the thin-shell was derived [4]; instability of the Cauchy horizon was discussed by non-linear analysis [5]; gravitational force between two test bodies was calculated [6–9]; black holes in the brane-world were discussed [10,11]; inflating brane solutions were constructed [12–14]. Relations to the AdS/CFT correspondence were also discussed [15]. In all of these works, AdS spacetime or its modifications play important roles.

Moreover, recently, cosmological solutions in this scenario were found [16–22]. In these solutions, the standard cosmology is restored in low energy, provided that a parameter μ in the solutions is small enough. If the parameter μ is not small enough, it affects cosmological evolution of our universe as dark radiation [19]. Hence, the parameter μ should be very small in order that the brane-world scenario should be consistent with nucleosynthesis [18]. On the other hand, in Ref. [23], it was shown that 5-dimensional geometry of all these cosmological solutions is the Schwarzschild-AdS (Sch-AdS) spacetime [24] and that μ is the mass parameter of the black hole. Therefore, the 5-dimensional bulk geometry should be the Sch-AdS spacetime with a small mass, which is close to the pure AdS spacetime. Moreover, black holes with small mass will evaporate in a short timescale [25]. Thus, it seems a good approximation to consider the pure AdS spacetime as a 5-dimensional bulk geometry for the brane-world cosmology.

Since the cosmological solution reproduces the standard cosmology as evolution of a homogeneous isotropic universe in low energy, this scenario may be considered as a realistic cosmology. Hence, it seems effective to look for observable consequences of this scenario. For this purpose, cosmic microwave background (CMB) anisotropy is a powerful tool. Therefore, we would like to give theoretical predictions of the brane-world scenario on the CMB anisotropy. However, this is not a trivial task as we shall explain below [1]. The main points are the following two: (i) how to give the

[1]A part of information about the CMB anisotropy can be derived from the conservation of energy momentum tensor [26].

initial condition; (ii) how to evolve perturbations. As for the first point, there is essentially the same issue even in the standard cosmology. The creation-from-nothing scenario may solve it [27–29] or may not.

Regarding the second point, we would like to argue that evolution of cosmological perturbations becomes non-local in the brane-world scenario. First, provided a suitable initial condition is given, perturbations localized on the brane can produce gravitational waves. Next, those gravitational waves propagate in the bulk AdS spacetime, and may collide with the brane at a spacetime point different from the spacetime point at which the gravitational waves were produced. When they collide with the brane, they should alter evolution of perturbations localized on the brane. Hence, evolution of perturbations localized on the brane should be non-local in the sense that it should be described by some integro-differential equations. Thus, the non-locality seems the essential point of evolution of cosmological perturbations in the brane world scenario. Without considering this point, we cannot expect drastic differences between CMB anisotropies predicted by the brane-world cosmology and the standard cosmology. Therefore, we have to consider the non-locality caused by gravitational waves seriously.

Towards the derivation of the integro-differential equations, it seems an important step to analyze gravitational perturbations in the bulk AdS geometry.

The purpose of this paper is to develop a gauge-invariant formalism of gravitational perturbations of maximally symmetric spacetimes including AdS spacetime. Existence of scalar-type master variables is shown, and the corresponding master equations are derived.

In Sec. II properties of the background spacetime are summarized. In Sec. III gauge-invariant variables are constructed. In Sec. IV linearized Einstein equation is expressed in terms of the gauge-invariant variables. In Sec. V existence of scalar-type master variables is shown and the corresponding master equations are derived. Sec. VI is devoted to a summary of this paper.

II. BACKGROUND SPACETIME

A spacetime is said to be maximally symmetric if it admits the maximum number $D(D+1)/2$ of independent Killing vector fields, where D is the dimensionality of the spacetime. It can be shown that a maximally symmetric spacetime is a spacetime of constant curvature and that it is uniquely specified by a curvature constant [30]. These are deSitter, Minkowski, and anti-deSitter spacetimes for positive, zero, and negative values of the curvature constant, respectively [31].

Since we consider a maximally symmetric spacetime as the background geometry and it is of constant curvature as mentioned above, we have the following equation for the background curvature tensor.

$$R^{(0)}_{MLNL'} = \frac{2\Lambda}{(D-1)(D-2)}(g^{(0)}_{MN}g^{(0)}_{LL'} - g^{(0)}_{ML'}g^{(0)}_{LN}), \tag{1}$$

where Λ is a constant called a cosmological constant, and the superscript (0) hereafter denotes that the quantity is calculated for the unperturbed spacetime. Note that the normalization in the right hand side is determined so that

$$G^{(0)}_{MN} + \Lambda g^{(0)}_{MN} = 0. \tag{2}$$

As the brane-world cosmology, in many cases of physical interests, the boundary of an unperturbed spacetime is a world-volume of a constant-curvature (or equivalently, maximally symmetric) subspace. Hence, it is convenient to decompose the background spacetime into a family of constant-curvature subspace: let us consider the decomposition

$$g^{(0)}_{MN} = \gamma_{ab}dx^a dx^b + r^2\Omega_{ij}dx^i dx^j, \tag{3}$$

where Ω_{ij} is a metric of a n-dimensional constant-curvature space, γ_{ab} is a $(D-n)$-dimensional metric depending only on the $(D-n)$-dimensional coordinates $\{x^a\}$, and r also depends only on $\{x^a\}$. Denoting the curvature constant of Ω_{ij} by K, we can write the curvature tensor of Ω_{ij} as

$$R^{(\Omega)ij}{}_{kl} = K(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k). \tag{4}$$

By using this expression, it is easy to show by explicit calculation that the curvature tensor of the background metric of the form (3) has the following components.

$$\begin{aligned}
R^{(0)ij}{}_{kl} &= \left(\frac{K}{r^2} - \gamma^{ab}\partial_a \ln r \partial_b \ln r\right)(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k), \\
R^{(0)i}{}_{ajb} &= -\delta^i_j(\nabla_a\nabla_b \ln r + \partial_a \ln r \partial_b \ln r), \\
R^{(0)}_{abcd} &= R^{(\gamma)}_{abcd},
\end{aligned} \tag{5}$$

where ∇_a is the covariant derivative compatible with the metric γ_{ab}, and $R^{(\gamma)}_{abcd}$ is the curvature tensor of γ_{ab}. Therefore, the condition (1) implies that

$$\begin{aligned}
\gamma^{ab}\partial_a \ln r \partial_b \ln r &= \frac{K}{r^2} - \frac{2\Lambda}{(D-1)(D-2)},\\
\nabla_a\nabla_b \ln r + \partial_a \ln r \partial_b \ln r &= -\frac{2\Lambda\gamma_{ab}}{(D-1)(D-2)},\\
R^{(\gamma)}_{abcd} &= \frac{2\Lambda}{(D-1)(D-2)}(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}).
\end{aligned} \tag{6}$$

These equations will be used repeatedly in this paper.

Now, we have three kinds of covariant derivatives for the background geometry: the first, which we shall denote by semicolons, is the covariant derivative compatible with the original D-dimensional background metric $g^{(0)}_{MN}$; secondly, ∇_a is the covariant derivative compatible with the $(D-n)$-dimensional metric γ_{ab}; and the third, which we shall denote by D_i, is compatible with Ω_{ij}. The following examples of relations among them can be easily obtained. First, for an arbitrary 1-form field V_M,

$$\begin{aligned}
V_{a;b} &= \nabla_b V_a,\\
V_{a;i} &= \partial_i V_a - V_i \partial_a \ln r,\\
V_{i;a} &= \partial_a V_i - V_i \partial_a \ln r,\\
V_{i;j} &= D_j V_i + r^2 \Omega_{ij}\gamma^{ab} V_a \partial_b \ln r.
\end{aligned} \tag{7}$$

These relations will be used when we seek infinitesimal gauge transformations for perturbations in Sec. III. The second set of examples is given in Appendix A and will be useful when we calculate Einstein equation for perturbations in Sec. IV.

III. GAUGE-INVARIANT VARIABLES

The main purpose of this paper is to derive equations governing gravitational perturbations around the background specified in the previous section. In other words, defining perturbation δg_{MN} by

$$g_{MN} = g^{(0)}_{MN} + \delta g_{MN}, \tag{8}$$

we shall derive the Einstein equation linearized with respect to δg_{MN}. Hence, in principle, independent variables are all components of δg_{MN}. However, those include degrees of freedom of gauge transformation as we shall see explicitly. Thus, it is desirable to reduce the number of degrees of freedom so that reduced degrees of freedom include physical perturbations only. For this purpose, in this section, we shall construct gauge invariant variables. The advantages of this approach against gauge-fixing will be explained in Sec. VI.

Since the background geometry still has the symmetry of isometry of Ω_{ij} even after the decomposition (3), it is convenient to expand perturbations by harmonics on the constant-curvature space:

$$\begin{aligned}
\delta g_{MN} dx^M dx^N = \sum_k \big[& h_{ab} Y dx^a dx^b + 2(h_{(T)a}V_{(T)i} + h_{(L)a}V_{(L)i})dx^a dx^i\\
&+ (h_{(T)}T_{(T)ij} + h_{(LT)}T_{(LT)ij} + h_{(LL)}T_{(LL)ij} + h_{(Y)}T_{(Y)ij})dx^i dx^j\big],
\end{aligned} \tag{9}$$

where Y, $V_{(T,L)}$ and $T_{(T,LT,LL,Y)}$ are scalar, vector and tensor harmonics, respectively, and the coefficients h_{ab}, $h_{(T,L)a}$ and $h_{(T,LT,LL,Y)}$ are supposed to depend only on the $(D-n)$-dimensional coordinates $\{x^a\}$. Hereafter, k denotes continuous ($K = 0, -1$) or discrete ($K = 1$) eigenvalues, and we omit them in most cases. In this respect, the summation with respect to k should be understood as an integration for $K = 0, -1$. See Appendix B for definitions and basic properties of the harmonics.

As mentioned already, δg_{MN} includes degrees of freedom of gauge transformation. In fact, an infinitesimal gauge transformation is given by

$$\delta g_{MN} \to \delta g_{MN} - \bar{\xi}_{M;N} - \bar{\xi}_{N;M}, \tag{10}$$

where $\bar{\xi}_M$ is an arbitrary vector field. Hence, by expanding the vector $\bar{\xi}_M$ in terms of harmonics as

$$\bar{\xi}_M dx^M = \sum_k \left[\xi_a Y dx^a + (\xi_{(T)} V_{(T)i} + \xi_{(L)} V_{(L)i}) dx^i\right], \tag{11}$$

we get the following infinitesimal gauge transformation for the expansion coefficients in Eq. (9).

$$\begin{aligned}
h_{ab} &\to h_{ab} - \nabla_a \xi_b - \nabla_b \xi_a, \\
h_{(T)a} &\to h_{(T)a} - r^2 \partial_a (r^{-2} \xi_{(T)}), \\
h_{(L)a} &\to h_{(L)a} - \xi_a - r^2 \partial_a (r^{-2} \xi_{(L)}), \\
h_{(T)} &\to h_{(T)}, \\
h_{(LT)} &\to h_{(LT)} - \xi_{(T)}, \\
h_{(LL)} &\to h_{(LL)} - \xi_{(L)}, \\
h_{(Y)} &\to h_{(Y)} - \gamma^{ab} \xi_a \partial_b r^2 + \frac{2k^2}{n} \xi_{(L)}.
\end{aligned} \tag{12}$$

Note that we have used Eq. (7) to derive those gauge transformations.

From the gauge transformations (12), it is easy to construct gauge invariant variables as follows.

$$\begin{aligned}
F_{ab} &= h_{ab} - \nabla_a X_b - \nabla_b X_a, \\
F &= h_{(Y)} - \gamma^{ab} X_a \partial_b r^2 + \frac{2k^2}{n} h_{(LL)}, \\
F_a &= h_{(T)a} - r^2 \partial_a (r^{-2} h_{(LT)}), \\
F_{(T)} &= h_{(T)},
\end{aligned} \tag{13}$$

where X_a is a gauge-dependent combination defined by

$$X_a = h_{(L)a} - r^2 \partial_a (r^{-2} h_{(LL)}), \tag{14}$$

and transforms under the infinitesimal gauge transformation as

$$X_a \to X_a - \xi_a. \tag{15}$$

Note that, for $k^2 = 0$, F_{ab} and F are not gauge invariant variables but gauge dependent variables since $V_{(L)i} \equiv 0$ and $T_{(LL)ij} \equiv 0$. Similarly, for a special value of k such that $T_{(LT)ij} \equiv 0$, F_a is not a gauge invariant variable but a gauge dependent variable. For all other values of k, off course, F_{ab}, F, F_a and $F_{(T)}$ are gauge invariant variables.

IV. EINSTEIN EQUATION FOR THE GAUGE-INVARIANT VARIABLES

In this section we shall seek linearized equations for the gauge invariant variables F_{ab}, F and $F_{(T)}$ constructed in the previous section. For this purpose, first, we expand the Einstein tensor in powers of δg_{MN} without using the expansion (9) nor any properties of the background geometry. The result is

$$G_{MN} = G^{(0)}_{MN} + G^{(1)}_{MN} + O(\delta g^2), \tag{16}$$

where

$$\begin{aligned}
2G^{(1)}_{MN} = &-\delta g_{MN;L}{}^{;L} + (\delta g^L{}_{M;LN} + \delta g^L{}_{N;LM}) - \delta g_{;MN} + (\delta g^{;L}{}_{;L} - \delta g^{LL'}{}_{;LL'}) g^{(0)}_{MN} \\
&+ (R^{(0)L}_M \delta g_{LN} + R^{(0)L}_N \delta g_{LM}) + g^{(0)}_{MN} R^{(0)}_{LL'} \delta g^{LL'} - R^{(0)} \delta g_{MN} - 2R^{(0)}_{MLNL'} \delta g^{LL'},
\end{aligned} \tag{17}$$

and $\delta g^{MN} \equiv g^{(0)MM'} g^{(0)NN'} \delta g_{M'N'}$, $\delta g \equiv g^{(0)MN} \delta g_{MN}$. Correspondingly, the linearized Einstein equation becomes

$$G^{(1)}_{MN} + \Lambda \delta g_{MN} = 0. \tag{18}$$

Next, because of the constant-curvature condition (1) for the background, the left hand side of Eq. (18) multiplied by 2 is rewritten as

$$2(G^{(1)}_{MN}+\Lambda\delta g_{MN}) = -\delta g_{MN;L}{}^{;L} + (\delta g^{L}{}_{M;LN} + \delta g^{L}{}_{N;LM}) - \delta g_{;MN} + (\delta g^{;L}{}_{;L} - \delta g^{LL'}{}_{;LL'})g^{(0)}_{MN}$$
$$+\frac{4\Lambda}{(D-1)(D-2)}\delta g_{MN} + \frac{2(D-3)\Lambda}{(D-1)(D-2)}\delta g\; g^{(0)}_{MN}. \tag{19}$$

Thirdly, by using the formulas (A2) given in Appendix A and substituting the expansion (9) into Eq. (19), we obtain the following expansion of the linearized Einstein equation.

$$2(G^{(1)}_{MN}+\Lambda\delta g_{MN})dx^M dx^N = \sum_k \left[E_{ab}Y dx^a dx^b + 2(E_{(T)a}V_{(T)i} + E_{(L)a}V_{(L)i})dx^a dx^i\right.$$
$$\left. +(E_{(T)}T_{(T)ij} + E_{(LT)}T_{(LT)ij} + E_{(LL)}T_{(LL)ij} + E_{(Y)}T_{(Y)ij})dx^i dx^j\right], \tag{20}$$

where the coefficients E_{ab}, $E_{(T,L)a}$ and $E_{(T,LT,LL,Y)}$ depend only on the $(D-n)$-dimensional coordinates $\{x^a\}$. Hereafter, let us call those coefficients *Einstein-equation-forms*. The linearized Einstein equation is equivalent to the following set of projected equations.

$$\begin{aligned} E_{ab} &= 0, \\ E_{(T)a} &= E_{(L)a} = 0, \\ E_{(T)} &= E_{(LT)} = E_{(LL)} = E_{(Y)} = 0. \end{aligned} \tag{21}$$

The next task is to express all Einstein-equation-forms in terms of the gauge invariant variables only. However, before showing the results, let us make classification of perturbations in order to make arguments clear.

Now, even without explicit expressions, it is easily shown from orthogonality between different kinds of harmonics (see Appendix B) that (i) E_{ab}, $E_{(L)a}$, $E_{(LL)}$ and $E_{(Y)}$ depend only on h_{ab}, $h_{(L)a}$, $h_{(LL)}$ and $h_{(Y)}$; (ii) $E_{(T)a}$ and $E_{(LT)}$ depend only on $h_{(T)a}$ and $h_{(LT)}$; (iii) $E_{(T)}$ depends only on $h_{(T)}$. Therefore, it is convenient to classify all perturbations into three categories: (i) (h_{ab}, $h_{(L)a}$, $h_{(LL)}$, $h_{(Y)}$); (ii) ($h_{(T)a}$, $h_{(LT)}$); (iii) $h_{(T)}$. It is evident that each category can be analyzed independently. Let us call perturbations in the first, second and third categories *scalar perturbations*, *vector perturbations* and *tensor perturbations*, respectively [2].

A. Einstein equation for scalar perturbations

For scalar perturbations given by

$$\delta g_{MN}dx^M dx^N = \sum_k \left[h_{ab}Y dx^a dx^b + 2h_{(L)a}V_{(L)i}dx^a dx^i + (h_{(LL)}T_{(LL)ij} + h_{(Y)}T_{(Y)ij})dx^i dx^j\right], \tag{22}$$

appropriate gauge invariant variables are F_{ab} and F, and appropriate Einstein-equation-forms are E_{ab}, $E_{(L)a}$, $E_{(LL)}$ and $E_{(Y)}$. The explicit expressions for the Einstein-equation-forms in terms of the gauge invariant variables are as follows.

$$\begin{aligned} E_{ab} = & -\nabla^2 F_{ab} + \nabla_a\nabla^c F_{cb} + \nabla_b\nabla^c F_{ca} - \nabla_a\nabla_b F^c_c \\ & +n(\nabla_a F^c_b + \nabla_b F^c_a - \nabla^c F_{ab})\partial_c \ln r + \left[\frac{k^2}{r^2} - \frac{4(n-1)\Lambda}{(D-1)(D-2)}\right]F_{ab} \\ & +\frac{n}{r^2}\left\{-\nabla_a\nabla_b F + \partial_a F\partial_b \ln r + \partial_b F\partial_a \ln r - 2F\partial_a \ln r\partial_b \ln r\right\} \\ & +\gamma_{ab}\left\{\nabla^2 F^c_c - \nabla^c\nabla^d F_{cd} + n(\nabla^d F^c_c - 2\nabla_c F^{cd})\partial_d \ln r\right. \\ & \left.+\left[\frac{2(D-3)\Lambda}{(D-1)(D-2)} - \frac{k^2}{r^2}\right]F^c_c - nF^{cd}(n\partial_c \ln r\partial_d \ln r + \nabla_c\nabla_d \ln r)\right\} \\ & +\frac{\gamma_{ab}}{r^2}\left\{n\nabla^2 F + n(n-3)\partial^c F\partial_c \ln r\right. \\ & \left.+F\left[-n(n-2)\partial^c \ln r\partial_c \ln r - n\nabla^2 \ln r + \frac{2(D-3)n\Lambda}{(D-1)(D-2)} - (n-1)\frac{k^2}{r^2}\right]\right\}, \end{aligned}$$

[2] This way of classification is the same as that adopted in the theory of cosmological perturbations [32]

$$
\begin{aligned}
E_{(L)a} &= r^{-(n-2)}\nabla^b(r^{n-2}F_{ab}) - (n-1)\partial_a(r^{-2}F) - r\partial_a(r^{-1}F^b_b),\\
E_{(LL)} &= -\frac{1}{2}[F^a_a + (n-2)r^{-2}F],\\
E_{(Y)} &= nr^2\left\{\nabla^2 F^a_a - \nabla^a\nabla^b F_{ab} - 2(n-1)\nabla_a F^{ab}\partial_b \ln r + (n-1)\partial^b F^a_a \partial_b \ln r\right.\\
&\quad -F^{ab}\left[(n^2-2n+2)\partial_a \ln r\partial_b \ln r + n\nabla_a\nabla_b \ln r\right] + F^a_a\left.\left[\frac{2(D-3)\Lambda}{(D-1)(D-2)} - \frac{n-1}{n}\frac{k^2}{r^2}\right]\right\}\\
&\quad +n\left\{(n-1)\nabla^2 F + (n-4)(n-1)\partial^a F\partial_a \ln r\right.\\
&\quad \left. +F\left[-(n^2-4n+2)\partial^a \ln r\partial_a \ln r - (n-2)\nabla^2 \ln r + \frac{2(Dn-3n+2)\Lambda}{(D-1)(D-2)} - \frac{(n-1)(n-2)}{n}\frac{k^2}{r^2}\right]\right\}.
\end{aligned} \tag{23}
$$

We have used the relations (6) to derive these expressions. Note that these are expressed in terms of gauge-invariant variables only, as expected.

Although each of the Einstein-equation-forms gives an equation for scalar perturbations, they do not give independent equations because of the Bianchi identity. In fact, the equation $E_{(Y)} = 0$ can be derived from $E_{(L)a} = 0$ and $E_{(LL)} = 0$. Thus, a set of independent equations of motion for scalar perturbations are given by $E_{ab} = 0$ and

$$
\begin{aligned}
F^a_a + (n-2)r^{-2}F = 0,\\
\nabla^b(r^{n-2}F_{ab}) = \partial_a(r^{n-4}F).
\end{aligned} \tag{24}
$$

Here we have rewritten $E_{(L)a} = 0$ into the form of the last equation by using $E_{(LL)} = 0$.

B. Einstein equation for vector perturbations

For vector perturbations given by

$$
\delta g_{MN}dx^M dx^N = \sum_k \left[2h_{(T)a}V_{(T)i}dx^a dx^i + h_{(LT)}T_{(LT)ij}dx^i dx^j\right], \tag{25}
$$

the appropriate gauge invariant variable is F_a, and appropriate Einstein-equation-forms are $E_{(T)a}$ and $E_{(LT)}$. The explicit expressions for the Einstein-equation-forms in terms of the gauge invariant variables are as follows.

$$
\begin{aligned}
E_{(T)a} &= -r^{-n}\nabla^b\left[r^{n+2}\nabla_b\left(\frac{F_a}{r^2}\right) - r^{n+2}\nabla_a\left(\frac{F_b}{r^2}\right)\right] + \frac{k^2-(n-1)K}{r^2}F_a,\\
E_{(LT)} &= r^{-(n-2)}\nabla^a(r^{n-2}F_a).
\end{aligned} \tag{26}
$$

We have used the relations (6) to derive these expressions.

C. Einstein equation for tensor perturbations

For tensor perturbations given by

$$
\delta g_{MN}dx^M dx^N = \sum_k h_{(T)}T_{(T)ij}dx^i dx^j, \tag{27}
$$

the coefficient $h_{(T)}$ itself is the gauge invariant variable $F_{(T)}$, and the appropriate Einstein-equation-form is $E_{(T)}$. The explicit expressions for the Einstein-equation-form is given by

$$
E_{(T)} = -r^{-(n-2)}\nabla^a[r^n\nabla_a(r^{-2}F_{(T)})] + \frac{k^2+2K}{r^2}F_{(T)}, \tag{28}
$$

or equivalently,

$$
E_{(T)} = -r^{D-n+1}\nabla^a[r^{-(2D-n-2)}\nabla_a(r^{D-3}F_{(T)})] + \frac{k^2+[(D-1)(n-2)-D(D-3)]K}{r^2}F_{(T)}. \tag{29}
$$

We have used the relations (6) to derive these expressions.

V. MASTER EQUATIONS

In the previous section we obtained equations of motion for the gauge invariant variables. These are described as scalars (F and $F_{(T)}$), vectors (F_a) and 2-tensors (F_{ab}) on the $(D-n)$-dimensional spacetime with the metric γ_{ab}. In the easiest case when $D-n=1$, those vectors and tensors have only one component, and thus they can trivially be treated on the same footing as scalars. However, in general, treatment of vectors and tensors is more complicated than scalars. In this section we show that, also in the case when $D-n=2$, those vector fields and tensor fields can be described by scalar fields called master variables.

Since we consider only the $D-n=2$ case in this section, without loss of generality, we can adopt the following form of the metric γ_{ab}.

$$\gamma_{ab}dx^a dx^b = -2e^{\phi}dx_+ dx_-, \tag{30}$$

where ϕ is a function of the coordinates x_+ and x_-. In this coordinate, the condition (6) is written as

$$\begin{aligned}
e^{-\phi}\frac{\partial_+ r\partial_- r}{r^2} &= \frac{\Lambda}{(D-1)(D-2)} - \frac{K}{2r^2},\\
\partial_+(e^{-\phi}\partial_+ r) &= 0,\\
\partial_-(e^{-\phi}\partial_- r) &= 0,\\
e^{-\phi}\frac{\partial_+\partial_- r}{r} &= \frac{2\Lambda}{(D-1)(D-2)},\\
e^{-\phi}\partial_+\partial_-\phi &= \frac{2\Lambda}{(D-1)(D-2)}
\end{aligned} \tag{31}$$

A. Master equation for scalar perturbations

Now let us show that the tensor F_{ab} and the scalar F can be described by one scalar variable if they satisfy Eqs. (24). First, by defining two scalars $\Phi_{(S)\pm}$ by

$$\begin{aligned}
\tilde{F}_{++} &= e^{\phi}\partial_+(e^{-\phi}\partial_+\Phi_{(S)+}),\\
\tilde{F}_{--} &= e^{\phi}\partial_-(e^{-\phi}\partial_-\Phi_{(S)-}),
\end{aligned} \tag{32}$$

Eqs. (24) can be rewritten as

$$e^{-\phi}\tilde{F}_{+-} = c\tilde{F}, \tag{33}$$

and

$$\begin{aligned}
\partial_+[(c+1)\tilde{F} + e^{-\phi}\partial_+\partial_-\Phi_{(S)+} - \tilde{\Lambda}\Phi_{(S)+}] &= 0,\\
\partial_-[(c+1)\tilde{F} + e^{-\phi}\partial_+\partial_-\Phi_{(S)-} - \tilde{\Lambda}\Phi_{(S)-}] &= 0,
\end{aligned} \tag{34}$$

where $\tilde{F}_{ab} = r^{D-4}F_{ab}$, $\tilde{F} = r^{D-6}F$, $c = (D-4)/2$ and $\tilde{\Lambda} = 2\Lambda/(D-1)(D-2)$. Eqs.(34) imply that there exist functions $f_1(x_-)$ and $f_2(x_+)$ such that

$$\begin{aligned}
(c+1)\tilde{F} + e^{-\phi}\partial_+\partial_-\Phi_{(S)+} - \tilde{\Lambda}\Phi_{(S)+} &= f_1(x_-),\\
(c+1)\tilde{F} + e^{-\phi}\partial_+\partial_-\Phi_{(S)-} - \tilde{\Lambda}\Phi_{(S)-} &= f_2(x_+).
\end{aligned} \tag{35}$$

Thus, consistency between these two equations requires that

$$e^{-\phi}\partial_+\partial_-(\Phi_{(S)+} - \Phi_{(S)-}) = \tilde{\Lambda}(\Phi_{(S)+} - \Phi_{(S)-}) + f_1(x_-) - f_2(x_+). \tag{36}$$

Next, let us solve the consistency condition (36) explicitly.

When $\tilde{\Lambda} = 0$, the last equation of (31) implies that there exist functions ϕ_+ and ϕ_- such that $\phi = \phi_+(x_+) + \phi_-(x_-)$. Thus, (36) can be solved easily to give

$$\Phi_{(S)+}-\Phi_{(S)-}=\bar{x}_+\int dx_- e^{\phi_-(x_-)}f_1(x_-)-\bar{x}_-\int dx_+ e^{\phi_+(x_+)}f_2(x_+)+f_3(x_-)-f_4(x_+), \tag{37}$$

where $\bar{x}_\pm=\int dx_\pm e^{\phi_\pm(x_\pm)}$, and f_3 and f_4 are arbitrary functions. Therefore, defining $\Phi_{(S)}$ by

$$\begin{aligned}\Phi_{(S)}&=\Phi_{(S)+}-\tilde{x}_+\int dx_- e^{\phi_-(x_-)}f_1(x_-)-f_3(x_-)\\&=\Phi_{(S)-}-\tilde{x}_-\int dx_+ e^{\phi_+(x_+)}f_2(x_+)-f_4(x_+),\end{aligned} \tag{38}$$

$\tilde{F}_{ab}$ and $\tilde{F}$ are written as

$$\begin{aligned}\tilde{F}_{++}&=e^{\phi}\partial_+(e^{-\phi}\partial_+\Phi_{(S)}),\\ \tilde{F}_{--}&=e^{\phi}\partial_-(e^{-\phi}\partial_-\Phi_{(S)}),\\ e^{-\phi}\tilde{F}_{+-}&=c\tilde{F}=-\frac{c}{c+1}e^{-\phi}\partial_+\partial_-\Phi_{(S)}.\end{aligned} \tag{39}$$

On the other hand, when $\tilde{\Lambda}\neq 0$, by defining Δ by $\Delta\equiv(\Phi_{(S)+}-\Phi_{(S)-})+(f_1(x_-)-f_2(x_+))/\tilde{\Lambda}$, the consistency condition (36) can be written as

$$\partial_+\partial_-\Delta=\tilde{\Lambda}e^{\phi}\Delta. \tag{40}$$

In Appendix C it is shown that a general solution of this equation is

$$\Delta=e^{-\phi}\partial_+(e^{\phi}C^+(x_+))+e^{-\phi}\partial_-(e^{\phi}C^-(x_-)), \tag{41}$$

where $C^\pm$ are arbitrary functions. Therefore, defining $\Phi_{(S)}$ by

$$\begin{aligned}\Phi_{(S)}&=\Phi_{(S)+}+f_1(x_-)/\tilde{\Lambda}-e^{-\phi}\partial_-(e^{\phi}C^-(x_-))\\&=\Phi_{(S)-}+f_2(x_+)/\tilde{\Lambda}+e^{-\phi}\partial_+(e^{\phi}C^+(x_+)),\end{aligned} \tag{42}$$

$\tilde{F}_{ab}$ and $\tilde{F}$ are written as

$$\begin{aligned}\tilde{F}_{++}&=e^{\phi}\partial_+(e^{-\phi}\partial_+\Phi_{(S)}),\\ \tilde{F}_{--}&=e^{\phi}\partial_-(e^{-\phi}\partial_-\Phi_{(S)}),\\ e^{-\phi}\tilde{F}_{+-}&=c\tilde{F}=\frac{c}{c+1}(-e^{-\phi}\partial_+\partial_-\Phi_{(S)}+\tilde{\Lambda}\Phi_{(S)}).\end{aligned} \tag{43}$$

In summary, for any value of $\tilde{\Lambda}$ there exists a master variable $\Phi_{(S)}$ such that $\tilde{F}_{ab}$ and $\tilde{F}$ are written as (43). These equations can be written covariantly as

$$\begin{aligned}r^{D-4}F_{ab}&=\nabla_a\nabla_b\Phi_{(S)}-\frac{D-3}{D-2}\nabla^2\Phi_{(S)}\gamma_{ab}-\frac{2(D-4)\Lambda}{(D-1)(D-2)^2}\Phi_{(S)}\gamma_{ab}\\ r^{D-6}F&=\frac{1}{D-2}\left[\nabla^2\Phi_{(S)}+\frac{4\Lambda}{(D-1)(D-2)}\Phi_{(S)}\right].\end{aligned} \tag{44}$$

Now, let us derive an equation of motion for the master variable $\Phi_{(S)}$. First, by substituting expressions (44) into the Einstein-equation-form E_{ab} given by (23), we can show that

$$r^{D-2}(E_{ab}-E^c_c\gamma_{ab})=\nabla_a\nabla_b\Delta_{(S)}+\frac{2\Lambda}{(D-1)(D-2)}\gamma_{ab}\Delta_{(S)}, \tag{45}$$

where

$$\Delta_{(S)}=r^2\left[-\nabla^2\Phi_{(S)}+(D-2)\partial^c\Phi_{(S)}\partial_c\ln r+\frac{2(D-4)\Lambda}{(D-1)(D-2)}\Phi_{(S)}\right]+[k^2-(D-2)K]\Phi_{(S)}. \tag{46}$$

Therefore, the projected Einstein equation $E_{ab}=0$ is equivalent to the statement that $\Delta_{(S)}$ is a solution of

$$\nabla_a \nabla_b \Delta_{(S)} + \frac{2\Lambda}{(D-1)(D-2)} \gamma_{ab} \Delta_{(S)} = 0. \tag{47}$$

Next, let us show that, by redefinition of $\Phi_{(S)}$, $\Delta_{(S)}$ can be set to be zero if $k^2[k^2 - (D-2)K] \neq 0$. Even in the case when $k^2 = 0$ and $K \neq 0$, $\Delta_{(S)}$ can be set to be of the form Cr, where C is a constant. The proof is as follows. It is easy to show by using (31) and (47) that

$$\begin{aligned} \partial_\pm \left(\frac{\Psi_1}{r} \right) &= 0, \\ \Psi_2 &= (D-2)Kr, \end{aligned} \tag{48}$$

where Ψ_1 and Ψ_2 are defined by

$$\begin{aligned} \Psi_1 &= r^2 \left[-\nabla^2 \Delta_{(S)} + (D-2) \partial^c \Delta_{(S)} \partial_c \ln r + \frac{2(D-4)\Lambda}{(D-1)(D-2)} \Delta_{(S)} \right], \\ \Psi_2 &= r^2 \left[-\nabla^2 r + (D-2) \partial^c r \partial_c \ln r + \frac{2(D-4)\Lambda}{(D-1)(D-2)} r \right] \end{aligned} \tag{49}$$

From the first equation of (48), $\Psi_1 = \tilde{C} r$, where $\tilde{C}$ is a constant. Therefore, if $k^2[k^2 - (D-2)K] \neq 0$ then

$$r^2 \left[-\nabla^2 \Phi'_{(s)} + (D-2) \partial^c \Phi'_{(s)} \partial_c \ln r + \frac{2(D-4)\Lambda}{(D-1)(D-2)} \Phi'_{(s)} \right] + [k^2 - (D-2)K] \Phi'_{(s)} = 0, \tag{50}$$

where

$$\Phi'_{(s)} = \Phi_{(S)} - \frac{\Delta_{(S)}}{k^2 - (D-2)K} + \frac{\tilde{C} r}{k^2[k^2 - (D-2)K]}. \tag{51}$$

When $k^2 = 0$ and $K \neq 0$, we can define $\Phi'_{(s)}$ by

$$\Phi'_{(s)} = \Phi_{(S)} + \frac{\Delta_{(S)}}{(D-2)K} \tag{52}$$

so that

$$r^2 \left[-\nabla^2 \Phi'_{(s)} + (D-2) \partial^c \Phi'_{(s)} \partial_c \ln r + \frac{2(D-4)\Lambda}{(D-1)(D-2)} \Phi'_{(s)} \right] - (D-2)K \Phi'_{(s)} = Cr, \tag{53}$$

where $C = \tilde{C}/(D-2)K$. It is evident from (31) and (47) that replacement of $\Phi_{(S)}$ with $\Phi'_{(s)}$ in Eq. (44) does not alter F_{ab} nor F.

Finally, F_{ab} and F are given by (44), where $\Phi_{(S)}$ is a solution of the master equation

$$\nabla^2 \Phi_{(S)} - (D-2) \partial^c \Phi_{(S)} \partial_c \ln r - \frac{2(D-4)\Lambda}{(D-1)(D-2)} \Phi_{(S)} - \frac{k^2 - (D-2)K}{r^2} \Phi_{(S)} + \frac{\Delta_{(S)}}{r^2} = 0. \tag{54}$$

When $k^2[k^2 - (D-2)K] \neq 0$, $\Delta_{(S)} = 0$. When $k^2 = 0$ and $K \neq 0$, $\Delta_{(S)} = Cr$, where C is an arbitrary constant. When $k^2 - (D-2)K = 0$, $\Delta_{(S)}$ is an arbitrary solution of (47).

B. Master equation for vector perturbations

Now let us consider vector perturbations. First, defining two functions $\Phi_{(V)\pm}$ by

$$r^{D-4} F_\pm = \pm \partial_\pm \Phi_{(V)\pm}, \tag{55}$$

$E_{(LT)} = 0$ is rewritten as

$$\partial_+ \partial_- (\Phi_{(V)+} - \Phi_{(V)-}) = 0, \tag{56}$$

where $E_{(LT)}$ is given by (26). Thus, there are functions $f_5(x_-)$ and $f_6(x_+)$ such that

$$\Phi_{(V)+} - \Phi_{(V)-} = f_5(x_-) - f_6(x_+). \tag{57}$$

and that we can define a master variable $\Phi_{(V)}$ by

$$\Phi_{(V)} = \Phi_{(V)+} - f_5(x_-) = \Phi_{(V)-} - f_6(x_+). \tag{58}$$

With this definition, $F_\pm$ are expressed as

$$r^{D-4} F_\pm = \pm \partial_\pm \Phi_{(V)}, \tag{59}$$

or covariantly,

$$r^{D-4} F_a = \epsilon_a{}^b \partial_b \Phi_{(V)}, \tag{60}$$

where ϵ_{ab} is the Levi-Civita tensor defined by

$$\epsilon_{01} = -\epsilon_{10} = \sqrt{|\det \gamma_{ab}|}, \quad \epsilon_{00} = \epsilon_{11} = 0. \tag{61}$$

Note that $\nabla_c \epsilon_{ab} = 0$.

Next, by substituting (60) into (26), we obtain

$$\epsilon_b{}^a E_{(T)a} = \nabla_b \left\{ r^D \nabla^a [r^{-(D-2)} \nabla_a \Phi_{(V)}] - [k^2 - (D-3)K] \Phi_{(V)} \right\}, \tag{62}$$

where we have used the identity $\epsilon_b{}^a \epsilon_{a'}{}^c = \gamma_{ba'} \gamma^{ac} - \delta^c_b \delta^a_{a'}$. Thus, the projected Einstein equation $E_{(T)a} = 0$ is equivalent to the following master equation.

$$r^{D-2} \nabla^a [r^{-(D-2)} \nabla_a \Phi_{(V)}] - \frac{k^2 - (D-3)K}{r^2} \Phi_{(V)} + \frac{\Delta_{(V)}}{r^2} = 0, \tag{63}$$

where $\Delta_{(V)}$ is a constant. Note that, when $k^2 \neq (D-3)K$, $\Delta_{(V)}$ can be set to be zero by redefinition of $\Phi_{(V)}$.

VI. SUMMARY AND DISCUSSION

In summary, we have investigated classical perturbations of D-dimensional maximally-symmetric spacetimes (Minkowski, deSitter, and anti-deSitter spacetimes). We have decomposed the background spacetime into a family of n-dimensional constant-curvature spaces and have expanded gravitational perturbations by harmonics on the constant-curvature space. After analyzing gauge transformation, we constructed gauge-invariant variables. Those can be considered as scalar fields F and $F_{(T)}$, the vector field F_a, and the symmetric second-rank tensor fields F_{ab} in $(D-n)$-dimensional spacetime.

When $D - n = 2$, we have shown that the tensor field F_{ab} and the vector field F_a can be described by scalar master variables. Namely, F_{ab} as well as F are given by (44), and F_a is given by (60). Therefore, in this case, we can investigate the gauge-invariant perturbations by analyzing the master scalar variables $\Phi_{(S)}$ and $\Phi_{(V)}$, and the scalar $F_{(T)}$. These scalar fields obey the following master equations.

$$r^{\alpha+\beta} \nabla^a [r^{-\alpha} \nabla_a (r^{-\beta} \Phi)] - (k^2 + \gamma K) r^{-2} \Phi + \Delta r^{-2} = 0, \tag{64}$$

where Φ represents $\Phi_{(S)}$, $\Phi_{(V)}$ or $F_{(T)}$, and K is the curvature constant of the $(D-2)$-dimensional constant-curvature space. The constants (α, β, γ) are given by Table I, and Δ is a constant or a function given by Table II.

In 4-dimension ($D = 4$), there is a choice such that $\alpha = 0$ for both $\Phi_{(S)}$ and $\Phi_{(V)}$, and there are no degrees of freedom of $F_{(T)}$ since $T_{(T)ij} \equiv 0$ for $n = 2$. (See the last paragraph of Appendix B.) Thus, the result of this paper is consistent with the master equations given in Refs. [33,34] for $D = 4$, $K = 1$.

Here, we mention again that, for $k^2 = 0$, F_{ab} and F are not gauge invariant variables but gauge dependent variables since $V_{(L)i} \equiv 0$ and $T_{(LL)ij} \equiv 0$. Similarly, for a special value of k such that $T_{(LT)ij} \equiv 0$, F_a is not a gauge invariant variable but a gauge dependent variable. The corresponding gauge transformations are

$$\begin{aligned} F_{ab} &\to F_{ab} - \nabla_a \xi_b - \nabla_b \xi_a \qquad (k^2 = 0), \\ F &\to F_{(Y)} - \gamma^{ab} \xi_a \partial_b r^2 \qquad (k^2 = 0), \end{aligned} \tag{65}$$

and

$$F_a \to F_a - r^2 \partial_a (r^{-2} \xi_{(T)}) \qquad (\text{for } k \text{ such that } T_{(LT)ij} \equiv 0). \tag{66}$$

The gauge transformation (65) for F_{ab} and F can be considered as the $(D-n)$-dimensional gauge transformation, provided that F_{ab} and F are considered as perturbations of γ_{ab} and r^2, respectively. On the other hand, since $T_{(LT)ij} \equiv 0$ implies that $V_{(T)i}$ is a Killing vector field of the metric Ω_{ij}, from general arguments of the Kaluza-Klein theory the vector field F_a $(= h_{(T)a})$ for such a value of k can be considered as a gauge field with the gauge group of the isometry of Ω_{ij}. Correspondingly, (66) can be considered as the gauge transformation of the gauge field. For all other values of k, off course, F_{ab}, F, F_a and $F_{(T)}$ are gauge invariant variables.

As already explained in Sec. I, this paper may be considered as the first step towards the derivation of the integro-differential equations for cosmological perturbations in the brane-world scenario. In this respect, the next step, which is now under investigation, is simplification of Israel's junction condition [35] for the master variables given in this paper.

Now let us discuss about a gauge choice which might be convenient in some cases. From the gauge transformations (12), by choosing ξ_a, $\xi_{(T)}$ and $\xi_{(L)}$ as

$$\begin{aligned} \xi_{(T)} &= h_{(LT)}, \\ \xi_{(L)} &= h_{(LL)}, \\ \xi_a &= h_{(L)a} - r^2 \partial_a (r^{-2} h_{(LL)}), \\ \xi_{(L)} &= -\frac{n}{2k^2} (h_{(Y)} - \gamma^{ab} h_{(L)a} \partial_b r^2) \qquad (\text{for } k (\neq 0) \text{ such that } T_{(LL)ij} \equiv 0). \end{aligned} \tag{67}$$

we can always make a gauge transformation such that

$$\begin{aligned} h_{(LT)} &\to 0, \\ h_{(LL)} &\to 0, \\ h_{(L)a} &\to 0, \\ h_{(Y)} &\to 0 \qquad (\text{for } k(\neq 0) \text{ such that } T_{(LL)ij} \equiv 0). \end{aligned} \tag{68}$$

This gauge choice was adopted in Ref. [36] for $K = 1$ in a different context and may be considered as a generalization of the so-called Regge-Wheeler gauge [37]. (For $n = 2$, $T_{(T)ij} \equiv 0$ and there is no degrees of freedom of $h_{(T)}$. See the last paragraph of Appendix B.) The remaining gauge transformation is equivalent to the $(D-n)$-dimensional gauge transformation

$$\begin{aligned} h_{ab} &\to h_{ab} - \nabla_a \xi_b - \nabla_b \xi_a \qquad (k^2 = 0), \\ h_{(Y)} &\to h_{(Y)} - \gamma^{ab} \xi_a \partial_b r^2 \qquad (k^2 = 0), \end{aligned} \tag{69}$$

and

$$h_{(T)a} \to h_{(T)a} - r^2 \partial_a (r^{-2} \xi_{(T)}) \qquad (\text{for } k \text{ such that } T_{(LT)ij} \equiv 0). \tag{70}$$

The vector field $h_{(T)a}$ for k such that $T_{(LT)ij} \equiv 0$ can be considered as a gauge field with the gauge group of the isometry of Ω_{ij}, and (70) can be considered as the gauge transformation of the gauge field.

Although the above generalized Regge-Wheeler gauge might be convenient for some purposes, it seems inconvenient to adopt it when we analyze perturbations of the brane world. In fact, in general the brane is not located at $r = R(t)$ in this gauge even if we assume that the trajectory of the brane is given by $r = R(t)$ in the unperturbed spacetime. In order to show this, first, let us adopt a Gaussian gauge in a neighborhood of the world volume of the brane for a moment. Next, let us perform an infinitesimal gauge transformation so that the transformed metric perturbation satisfies the generalized Regge-Wheeler gauge. The infinitesimal gauge transformation is given by Eq. (67), provided that h's in the right hand side are calculated in the Gaussian gauge. Thus, it is easily seen that

$$\xi_w = -r^2 \partial_w (r^{-2} h_{(LL)}), \tag{71}$$

where $h_{(LL)}$ in the right hand side is calculated in the Gaussian gauge, and w is a coordinate corresponding to the geodesic distance from the brane. Therefore, the displacement of the brane in the generalized Regge-Wheeler gauge (ξ_w estimated at the brane) is not zero in general. (cf. Ref. [6]) In this respect, is is not convenient to adopt the generalized Regge-Wheeler gauge for the brane world: when one considers spacetime with singularities such as

a domain wall or the brane, perturbative treatment as well as the variational principle may break down unless the displacement of the singularities vanishes [33,38]. Therefore, if we would prefer to gauge-fixing method rather than the gauge-invariant formalism, we have to modify the generalized Regge-Wheeler gauge slightly so that the displacement of the brane vanishes. Otherwise, we have to introduce degrees of freedom for the displacement of the brane explicitly, and have to consider the consistent gauge transformation of it so that the gauge transformation does not change the physical position of the brane. Although the modification of the generalized Regge-Wheeler gauge may be achieved by allowing non-zero value of $h_{(L)a}$. it seems that in this modified gauge the analysis becomes complicated. Therefore, the gauge-invariant formalism developed in this paper seems better than gauge-fixing method for the analysis of perturbations of the brane world.

Finally, we suggest a possible generalization of the formalism developed in this paper. It seems possible to generalize the formalism to more general background spacetimes. In particular, generalization to Schwarzschild-AdS spacetime [24] is of physical interests since bulk geometry of cosmological solutions in the brane-world scenario are Schwarzschild-AdS spacetime in general [23].

ACKNOWLEDGMENTS

The author would like to thank Professor W. Israel for continuing encouragement. He would be grateful to Dr. T. Shiromizu and Professor M. Sasaki for discussions. This work was supported by the CITA National Fellowship and the NSERC operating research grant.

APPENDIX A: RELATIONS AMONG THREE KINDS OF COVARIANT DERIVATIVES

For an arbitrary (not necessarily symmetric) 2-tensor T_{MN},

$$\begin{aligned}
T_{ab;cd} &= \nabla_d\nabla_c T_{ab},\\
T_{ab;ci} &= \nabla_c\partial_i T_{ab} - \partial_i T_{ab}\partial_c \ln r - (\nabla_c T_{ib} - 2T_{ib}\partial_c \ln r)\partial_a \ln r - (\nabla_c T_{ai} - 2T_{ai}\partial_c \ln r)\partial_b \ln r,\\
T_{ab;ic} &= T_{ab;ci} - (T_{ai}\nabla_c\nabla_b \ln r + T_{ib}\nabla_c\nabla_a \ln r),\\
T_{ai;bc} &= \nabla_c\nabla_b T_{ai} - \nabla_b T_{ai}\partial_c \ln r - (\nabla_c T_{ai} - T_{ai}\partial_c \ln r)\partial_b \ln r - T_{ai}\nabla_c\nabla_b \ln r,\\
T_{ab;ij} &= r^2\Omega_{ij}[\nabla_c T_{ab} - T_{ac}\partial_b \ln r - T_{cb}\partial_a \ln r]\partial^c \ln r + D_j D_i T_{ab}\\
&\quad -(D_i T_{aj} + D_j T_{ai})\partial_b \ln r - (D_i T_{jb} + D_j T_{ib})\partial_a \ln r + (T_{ij} + T_{ji})\partial_a \ln r\partial_b \ln r,\\
T_{ai;bj} &= r^2\Omega_{ij}(\nabla_b T_{ac} - T_{ac}\partial_b \ln r)\partial^c \ln r + \nabla_b D_j T_{ai} - 2D_j T_{ai}\partial_b \ln r - (\partial_b T_{ji} - 3T_{ji}\partial_b \ln r)\partial_a \ln r,\\
T_{ai;jb} &= r^2\Omega_{ij}(\nabla_b T_{ac}\partial^c \ln r + T_{ac}\nabla_b\nabla^c \ln r) + \nabla_b D_j T_{ai} - 2D_j T_{ai}\partial_b \ln r\\
&\quad -(\partial_b T_{ji} - 2T_{ji}\partial_b \ln r)\partial_a \ln r - T_{ji}\nabla_b\nabla_a \ln r,\\
T_{ij;ab} &= \nabla_b\nabla_a T_{ij} - 2\partial_a T_{ij}\partial_b \ln r - 2\partial_b T_{ij}\partial_a \ln r + 2T_{ij}(2\partial_a \ln r\partial_b \ln r - \nabla_b\nabla_a \ln r),\\
T_{ij;ka} &= r^2\Omega_{ki}[\nabla_a T_{bj}\partial^b \ln r + T_{bj}(\nabla_a\nabla^b \ln r - \partial_a \ln r\partial^b \ln r)]\\
&\quad +r^2\Omega_{jk}[\nabla_a T_{ib}\partial^b \ln r + T_{ib}(\nabla_a\nabla^b \ln r - \partial_a \ln r\partial^b \ln r)] + \partial_a D_k T_{ij} - 3D_k T_{ij}\partial_a \ln r,\\
T_{ij;ak} &= r^2\Omega_{ik}(\nabla_a T_{bj}\partial^b \ln r - 2T_{bj}\partial_a \ln r\partial^b \ln r) + r^2\Omega_{jk}(\nabla_a T_{ib}\partial^b \ln r - 2T_{ib}\partial_a \ln r\partial^b \ln r)\\
&\quad +\partial_a D_k T_{ij} - 3D_k T_{ij}\partial_a \ln r,\\
T_{ai;jk} &= r^2(\Omega_{ik}\partial_j T_{ab} + \Omega_{ij}\partial_k T_{ab})\partial^b \ln r + r^2\Omega_{jk}(\nabla_b T_{ai} - T_{ai}\partial_b \ln r)\partial^b \ln r - r^2(\Omega_{jk}T_{bi} + \Omega_{ij}T_{kb})\partial_a \ln r\partial^b \ln r\\
&\quad -r^2\Omega_{ik}(T_{aj}\partial_b \ln r + T_{jb}\partial_a \ln r)\partial^b \ln r + D_k D_j T_{ai} - (D_k T_{ji} + D_j T_{ki})\partial_a \ln r,\\
T_{ij;kl} &= r^4(\Omega_{ik}\Omega_{jl} + \Omega_{il}\Omega_{jk})T_{ab}\partial^a \ln r\partial^b \ln r + r^2(\Omega_{ik}D_l T_{aj} + \Omega_{jl}D_k T_{ia} + \Omega_{jk}D_l T_{ia} + \Omega_{il}D_k T_{aj})\partial^a \ln r\\
&\quad +r^2\Omega_{kl}(\partial_a T_{ij} - 2T_{ij}\partial_a \ln r)\partial^a \ln r - r^2(\Omega_{il}T_{kj} + \Omega_{jl}T_{ik})\partial_a \ln r\partial^a \ln r + D_l D_k T_{ij}.
\end{aligned} \tag{A1}$$

By using these, we can show the following equations, which are useful in Sec. IV.

$$\begin{aligned}
&-T_{ab}{}^{;L}{}_{;L} + T_a{}^L{}_{;Lb} + T_b{}^L{}_{;La} - T^L_{L;ab}\\
&\quad = -\nabla^2 T_{ab} + \nabla_a\nabla^c T_{cb} + \nabla_b\nabla^c T_{ca} - \nabla_a\nabla_b(T_{cd}\gamma^{cd}) + n(\nabla_a T_{bc} + \nabla_b T_{ac} - \nabla_c T_{ab})\partial^c \ln r\\
&\quad\quad +n(T_{ac}\partial_b \ln r + T_{cb}\partial_a \ln r)\partial^c \ln r + n(T_{ac}\nabla_b\nabla^c \ln r + T_{bc}\nabla_a\nabla^c \ln r) - r^{-2}D^2 T_{ab} + r^{-2}(\nabla_a D^i T_{bi} + \nabla_b D^i T_{ai})\\
&\quad\quad -r^{-2}\nabla_a\nabla_b(\Omega^{ij}T_{ij}) + r^{-2}[\partial_a(\Omega^{ij}T_{ij})\partial_b \ln r + \partial_b(\Omega^{ij}T_{ij})\partial_a \ln r] - 2r^{-2}(\Omega^{ij}T_{ij})\partial_a \ln r\partial_b \ln r
\end{aligned}$$

$$
\begin{aligned}
&-T_{ai}{}^{;L}{}_{;L} + T_a{}^L{}_{;Li} + T_i{}^L{}_{;La} - T^L_{L;ai} \\
&\quad = \nabla^b \partial_i T_{ab} - \partial_a \partial_i (T_{bc}\gamma^{bc}) + (n-2)\partial_i T_{ab}\partial^b \ln r + \partial_i (T_{bc}\gamma^{bc})\partial_a \ln r - \nabla^2 T_{ai} + \nabla_a \nabla^b T_{ib} \\
&\qquad + \nabla_a[(n-1)T_{ib} + T_{bi}]\partial^b \ln r - (n-2)\nabla^b T_{ai}\partial_b \ln r - 2\nabla^b T_{ib}\partial_a \ln r + T_{ai}(n\partial^b \ln r \partial_b \ln r + \nabla^2 \ln r) \\
&\qquad + [(n-1)T_{ib} + T_{bi}]\nabla_a \nabla^b \ln r - (n-2)(2T_{ib} - T_{bi})\partial_a \ln r \partial^b \ln r + r^{-2} D_i D^j T_{aj} - r^{-2} D^2 T_{ai} \\
&\qquad + r^{-2}\partial_a D^j T_{ij} - r^{-2}\partial_a \partial_i (\Omega^{jk} T_{jk}) - 2r^{-2} D^j (2T_{ij} - T_{ji})\partial_a \ln r + 2r^{-2}\partial_i (\Omega^{jk} T_{jk})\partial_a \ln r \\
&-T_{ij}{}^{;L}{}_{;L} + T_i{}^L{}_{;Lj} + T_j{}^L{}_{;Li} - T^L_{L;ij} \\
&\quad = r^2 \Omega_{ij}\left[2\nabla^a T_{ba}\partial^b \ln r - \partial_a (T_{bc}\gamma^{bc})\partial^a \ln r + 2(n-1)T_{ab}\partial^a \ln r \partial^b \ln r\right] - D_i D_j (T_{ab}\gamma^{ab}) \\
&\qquad + \nabla^a (D_i T_{ja} + D_j T_{ia}) + \{D_i[(n-1)T_{ja} - T_{aj}] + D_j[(n-3)T_{ia} + T_{ai}]\}\partial^a \ln r + 2\Omega_{ij} D^k T_{ak}\partial^a \ln r \\
&\qquad - \nabla^2 T_{ij} - (n-4)\partial_a T_{ij}\partial^a \ln r - \Omega_{ij}\partial_a (T_{kl}\Omega^{kl})\partial^a \ln r + 2[(n-1)\partial^a \ln r \partial_a \ln r + \nabla^2 \ln r]T_{ij} \\
&\qquad - r^{-2} D^2 T_{ij} + r^{-2}(D_i D^k T_{jk} + D_j D^k T_{ik}) - r^{-2} D_j D_i (T_{kl}\Omega^{kl}), \\
&T^L_L{}^{;L'}{}_{;L'} - T^{LL'}{}_{;LL'} \\
&\quad = \nabla^2 (T_{ab}\gamma^{ab}) - \nabla^b \nabla^a T_{ab} + n[\partial_a (T_{bc}\gamma^{bc}) - \nabla^b (T_{ab} + T_{ba})]\partial^a \ln r + r^{-2} D^2 (T_{ab}\gamma^{ab}) \\
&\qquad - nT_{ab}(n\partial^a \ln r \partial^b \ln r + \nabla^a \nabla^b \ln r) - r^{-2}\nabla^a D^i (T_{ai} + T_{ia}) - (n-1)r^{-2} D^i (T_{ai} + T_{ia})\partial^a \ln r \\
&\qquad + r^{-2}\nabla^2 (T_{ij}\Omega^{ij}) + (n-3)r^{-2}\partial_a (T_{ij}\Omega^{ij})\partial^a \ln r - r^{-2}(T_{ij}\Omega^{ij})[(n-2)\partial^a \ln r \partial_a \ln r + \nabla^2 \ln r] \\
&\qquad + r^{-4} D^2 (T_{ij}\Omega^{ij}) - r^{-4} D^i D^j T_{ji}.
\end{aligned}
\tag{A2}
$$

APPENDIX B: HARMONICS ON CONSTANT-CURVATURE SPACE

In this Appendix we give definitions and basic properties of scalar, vector and tensor harmonics on a n-dimensional constant-curvature space. Throughout this Appendix we will use the notation that Ω_{ij} is the metric of the constant-curvature space and that D_i is the covariant derivative compatible with Ω_{ij}. The curvature tensor of the space is given by Eq. (4).

1. scalar harmonics

The scalar harmonics is supposed to satisfy the following relations.

$$
\begin{aligned}
&D^2 Y + k^2 Y = 0, \\
&\int d^n x \sqrt{\Omega} Y Y = \delta.
\end{aligned}
\tag{B1}
$$

Hereafter, k^2 denotes continuous eigenvalues for $K = 0, -1$ [39] or discrete eigenvalues $k_l^2 = l(l+n-1)$ $(l = 0, 1, \cdots)$ for $K = 1$ [40], and we omit them in most cases. In this respect, the delta δ in equations above and below represents Dirac's delta function $\delta^n(k - k')$ for continuous eigenvalues and Kronecker's delta $\delta_{ll'}\delta_{mm'}$ for discrete eigenvalues, where m (and m') denotes a set of integers. Correspondingly, in the following arguments, a summation with respect to k should be understood as integration for $K = 0, -1$.

2. vector harmonics

First, in general, a vector field v_i can be decomposed as

$$
v_i = v_{(T)i} + \partial_i f, \tag{B2}
$$

where f is a function and $v_{(T)}$ is a transverse vector field:

$$
D^i v_{(T)i} = 0. \tag{B3}
$$

Thus, the vector field v_i can be expanded by using the scalar harmonics Y and transverse vector harmonics $V_{(T)i}$ as

$$V_i = \sum_k \left[c_{(T)} V_{(T)i} + c_{(L)} \partial_i Y \right], \tag{B4}$$

where $c_{(T)}$ and $c_{(L)}$ are constants depending on k, and the transverse vector harmonics $V_{(T)i}$ is supposed to satisfy the following relations.

$$\begin{aligned} D^2 V_{(T)i} + k^2 V_{(T)i} &= 0, \\ D^i V_{(T)i} &= 0, \\ \int d^n x \sqrt{\Omega} \Omega^{ij} V_{(T)i} V_{(T)j} &= \delta, \end{aligned} \tag{B5}$$

where k^2 denotes continuous eigenvalues for $K = 0, -1$ or discrete eigenvalues $k_l^2 = l(l+n-1) - 1$ $(l = 1, 2, \cdots)$ for $K = 1$ [40], and we omit them in most cases. From Eq. (B4), it is convenient to define longitudinal vector harmonics $V_{(L)i}$ by

$$V_{(L)i} \equiv \partial_i Y. \tag{B6}$$

It is easily shown that the longitudinal vector harmonics has the following properties.

$$\begin{aligned} D^2 V_{(L)i} + [k^2 - (n-1)K] V_{(L)i} &= 0, \\ D^i V_{(L)i} &= -k^2 Y, \\ D_{[i}(V_{(L)j]} &= 0, \\ \int d^n x \sqrt{\Omega} \Omega^{ij} V_{(L)i} V_{(L)j} &= k^2 \delta, \\ \int d^n x \sqrt{\Omega} \Omega^{ij} V_{(T)i} V_{(L)j} &= 0. \end{aligned} \tag{B7}$$

3. Tensor harmonics

First, in general, a symmetric second-rank tensor field t_{ij} can be decomposed as

$$t_{ij} = t_{(T)ij} + D_i v_j + D_j v_i + f \Omega_{ij}, \tag{B8}$$

where f is a function, v_i is a vector field and $t_{(T)ij}$ is a transverse traceless symmetric tensor field:

$$\begin{aligned} t^i_{(T)i} &= 0, \\ D^i t_{(T)ij} &= 0. \end{aligned} \tag{B9}$$

Thus, the tensor field t_{ij} can be expanded by using the vector harmonics $V_{(T)}$ and $V_{(L)}$, and transverse traceless tensor harmonics $T_{(T)}$ as

$$t_{ij} = \sum_k \left[c_{(T)} T_{(T)ij} + c_{(LT)} (D_i V_{(T)j} + D_j V_{(T)i}) + c_{(LL)} (D_i V_{(L)j} + D_j V_{(L)i}) + c_{(Y)} Y \Omega_{ij} \right], \tag{B10}$$

where $c_{(T)}$, $c_{(LT)}$, $c_{(LL)}$ and $c_{(Y)}$ are constants depending on k, and the transverse tensor harmonics $T_{(T)}$ is supposed to satisfy the following relations.

$$\begin{aligned} D^2 T_{(T)ij} + k^2 T_{(T)ij} &= 0, \\ T^i_{(T)i} &= 0, \\ D^i T_{(T)ij} &= 0, \\ \int d^n x \sqrt{\Omega} \Omega^{ik} \Omega^{jl} T_{(T)ij} T_{(T)kl} &= \delta, \end{aligned} \tag{B11}$$

where k^2 denotes continuous eigenvalues for $K = 0, -1$ or discrete eigenvalues $k_l^2 = l(l+n-1) - 2$ $(l = 2, 3, \cdots)$ for $K = 1$ [40], and we omit them in most cases. From Eq. (B10), it is convenient to define tensor harmonics $T_{(LT)}$, $T_{(LL)}$, and $T_{(Y)}$ by

$$
\begin{aligned}
T_{(LT)ij} &\equiv D_i V_{(T)j} + D_j V_{(T)i}, \\
T_{(LL)ij} &\equiv D_i V_{(L)j} + D_j V_{(L)i} - \frac{2}{n}\Omega_{ij} D^k V_{(L)k} \\
&= 2 D_i D_j Y + \frac{2}{n} k^2 \Omega_{ij} Y, \\
T_{(Y)ij} &\equiv \Omega_{ij} Y.
\end{aligned} \tag{B12}
$$

It is easily shown that these tensor harmonics satisfy the following properties.

$$
\begin{aligned}
D^2 T_{(LT)ij} + [k^2 - (n+1)K] T_{(LT)ij} &= 0, \\
D^i T_{(LT)ij} &= -[k^2 - (n-1)K] V_{(T)j}, \\
T^i_{(LT)i} &= 0,
\end{aligned} \tag{B13}
$$

$$
\begin{aligned}
D^2 T_{(LL)ij} + [k^2 - 2nK] T_{(LL)ij} &= 0, \\
D^i T_{(LL)ij} &= -\frac{2(n-1)}{n}(k^2 - nK) V_{(L)j}, \\
T^i_{(LL)i} &= 0,
\end{aligned} \tag{B14}
$$

and

$$
\begin{aligned}
D^2 T_{(Y)ij} + k^2 T_{(Y)ij} &= 0, \\
D^i T_{(Y)ij} &= V_{(L)j}, \\
T^i_{(Y)i} &= nY.
\end{aligned} \tag{B15}
$$

It is also easy to show the following formulas of integral as well as the orthogonality between any different types of tensor harmonics.

$$
\begin{aligned}
\int d^n x \sqrt{\Omega}\, \Omega^{ik}\Omega^{jl} T_{(LT)ij} T_{(LT)kl} &= 2[k^2 - (n-1)K]\delta, \\
\int d^n x \sqrt{\Omega}\, \Omega^{ik}\Omega^{jl} T_{(LL)ij} T_{(LL)kl} &= \frac{4(n-1)}{n}(k^2 - nK) k^2 \delta, \\
\int d^n x \sqrt{\Omega}\, \Omega^{ik}\Omega^{jl} T_{(Y)ij} T_{(Y)kl} &= n\delta.
\end{aligned} \tag{B16}
$$

Finally, we prove that $T_{(T)ij} \equiv 0$ for $n = 2$. First, without loss of generality, we can assume that the metric is of the form

$$
\Omega_{ij} dx^i dx^j = 2 e^{\psi} dz d\bar{z}, \tag{B17}
$$

where ψ is a function of a complex coordinate z and its complex conjugate $\bar{z}$. In this coordinate system, the transverse-traceless condition (B9) becomes

$$
\begin{aligned}
t_{(T)z\bar{z}} &= 0, \\
\partial_{\bar{z}} t_{(T)zz} = \partial_z t_{(T)\bar{z}\bar{z}} &= 0.
\end{aligned} \tag{B18}
$$

The second equation can be solved to give $t_{(T)zz} = t(z)$ and $t_{(T)\bar{z}\bar{z}} = \tilde{t}(\bar{z})$, where t and $\bar{t}$ are arbitrary holomorphic and anti-holomorphic functions. Thus, t_{ij} can be written as follows.

$$
t_{(T)ij} = D_i v_j + D_j v_i + f\Omega_{ij}, \tag{B19}
$$

where the vector v_i and the scalar f are defined by $v_z \equiv e^{-\psi}\int dz e^{\psi} t(z)/2$, $v_{\bar{z}} \equiv e^{-\psi}\int d\bar{z} e^{\psi}\tilde{t}(\bar{z})/2$ and $f = -e^{-\psi}(\partial_z v_{\bar{z}} + \partial_{\bar{z}} v_z)$. This means that any transverse-traceless tensor can be written in terms of $T_{(LT)ij}$, $T_{(LL)ij}$ and $T_{(Y)ij}$. This completes the proof that $T_{(T)ij} \equiv 0$ for $n = 2$.

APPENDIX C: GENERAL SOLUTION OF THE CONSISTENCY CONDITION

In this appendix, we seek a general solution of (40) for $\tilde{\Lambda} \neq 0$.

First, let us define a new function X by

$$\Delta = e^{-\phi}\partial_+(e^{\phi}X). \tag{C1}$$

Thence, the equation (40) is equivalent to

$$\partial_+[e^{-\phi}\partial_+(e^{\phi}\partial_- X)] = 0. \tag{C2}$$

This can be easily integrated to give

$$\begin{aligned} \partial_+(e^{\phi}\partial_- X) &= f_7(x_-)e^{\phi} \\ &= \frac{1}{\tilde{\Lambda}} f_7(x_-)\partial_+\partial_-\phi, \end{aligned} \tag{C3}$$

where $f_7(x_-)$ is an arbitrary function and we have used the last equation of (31) to obtain the last line. This equation can also be integrated to give

$$\begin{aligned} \partial_- X &= e^{-\phi}\left[\frac{1}{\tilde{\Lambda}} f_7(x_-)\partial_-\phi + f_8(x_-)\right] \\ &= -\frac{1}{\tilde{\Lambda}}\partial_-[e^{-\phi}f_7(x_-)] + e^{-\phi}\left[\frac{1}{\tilde{\Lambda}}\partial_- f_7(x_-) + f_8(x_-)\right], \end{aligned} \tag{C4}$$

where $f_8(x_-)$ is also an arbitrary function. Hence,

$$X = -\frac{1}{\tilde{\Lambda}} e^{-\phi} f_7(x_-) + \int dx_- e^{-\phi}\left[\frac{1}{\tilde{\Lambda}}\partial_- f_7(x_-) + f_8(x_-)\right] + f_9(x_+), \tag{C5}$$

where $f_9(x_+)$ is an arbitrary function. Therefore the general solution of (40) can be written as

$$\Delta = \Delta_+ + \Delta_-, \tag{C6}$$

where

$$\begin{aligned} \Delta_+ &= e^{-\phi}\partial_+[e^{\phi}C^+(x_+)], \\ \Delta_- &= e^{-\phi}\partial_+\left[e^{\phi}\int dx_- e^{-\phi}C(x_-)\right], \end{aligned} \tag{C7}$$

where C^+ and C are arbitrary functions.

Next, let us show that Δ_- can be rewritten as $e^{-\phi}\partial_-[e^{\phi}C^-(x_-)]$ by some function C^-. This is easily done in a particular coordinate system as we shall show below. Hence, let us see that the form of Δ_- and that of $e^{-\phi}\partial_-[e^{\phi}C^-(x_-)]$ are invariant under a coordinate transformation [3]. In fact, under a general coordinate transformation $x_\pm \to \tilde{x}_\pm = \tilde{x}_\pm(x_\pm)$ between double-null coordinate systems, these forms are invariant:

$$\begin{aligned} e^{-\phi}\partial_+\left[e^{\phi}\int dx_- e^{-\phi}C(x_-)\right] &= e^{-\tilde{\phi}}\tilde{\partial}_+\left[e^{\tilde{\phi}}\int d\tilde{x}_- e^{-\tilde{\phi}}\tilde{C}(\tilde{x}_-)\right], \\ e^{-\phi}\partial_-[e^{\phi}C^-(x_-)] &= e^{-\tilde{\phi}}\tilde{\partial}_-[e^{\tilde{\phi}}\tilde{C}^-(\tilde{x}_-)], \end{aligned} \tag{C8}$$

where $e^{\tilde{\phi}} = e^{\phi}(dx_+/d\tilde{x}_+)(dx_-/d\tilde{x}_-)$, $\tilde{\partial}_\pm = (\partial/\partial\tilde{x}_\pm)_{\tilde{x}_\mp}$ and

---

[3] Thanks to the 2-dimensional version of the uniqueness of the constant-curvature spacetime [30] (cf. the last equation of (6)), the metric γ_{ab} for different value of K can also be obtained by a coordinate transformation from the metric γ_{ab} in a particular coordinate system for a particular value of K, provided that Λ is common. However, the explicit expression of r will change if γ_{ab} is expressed in the common form.

$$\tilde{C}(\tilde{x}_-) = C(x_-)\left(\frac{dx_-}{d\tilde{x}_-}\right)^2,$$
$$\tilde{C}^-(\tilde{x}_-) = C^-(x_-)\frac{d\tilde{x}_-}{dx_-}. \tag{C9}$$

Now let us show in a particular coordinate system that Δ_- can actually be rewritten as $e^{-\phi}\partial_-[e^{\phi}C^-(x_-)]$ by some function C^-. For this purpose, it seems the easiest to consider a coordinate system in which

$$e^{\phi} = -\frac{(D-1)(D-2)}{\Lambda(x_+ - x_-)^2}. \tag{C10}$$

In this coordinate, it is easy to show by integrations by part that

$$\Delta_- = e^{-\phi}\partial_-[e^{\phi}C^-(x_-)], \tag{C11}$$

where $C^-(x_-)$ is defined by

$$\partial_-^3 C^-(x_-) = \frac{2\Lambda}{(D-1)(D-2)}C(x_-). \tag{C12}$$

Finally, we have shown that a general solution of (40) for $\tilde{\Lambda} \neq 0$ is

$$\Delta = e^{-\phi}\partial_+[e^{\phi}C^+(x_+)] + e^{-\phi}\partial_-[e^{\phi}C^-(x_-)], \tag{C13}$$

where $C^{\pm}$ are arbitrary functions.

---

[1] J. Maldacena, Adv. Theor. Math. Phys., **2**, 231(1998).
[2] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, hep-th/9905111 and reference therein.
[3] L. Randall and R. Sundrum, Phys.Rev.Lett. **83**, 4690 (1999) [hep-th/9906064].
[4] T. Shiromizu, K. Maeda and M. Sasaki, gr-qc/9910076.
[5] A. Chamblin and G. W. Gibbons, hep-th/9909130.
[6] J. Garriga and T. Tanaka, hep-th/9911055.
[7] T. Tanaka and X. Montes, hep-th/0001092.
[8] M. Sasaki, T. Shiromizu and K. Maeda, hep-th/9912233.
[9] S. B. Giddings, E. Katz, and L. Randall, hep-th/0002091.
[10] A. Chamblin, S. W. Hawking, and H. S. Reall, hep-th/9909205.
[11] R. Emparan, G. T. Horowitz and R. C. Myers, hep-th/9911043; hep-th/9912135.
[12] T. Nihei, Phys. Lett. **B465** 81 (1999) [hep-ph/9905487].
[13] N. Kaloper, Phys. Rev. **D60** 123506 (1999) [hep-th/9905210].
[14] H. B. Kim and H. D. Kim, hep-th/9909053.
[15] S. S. Gubser, hep-th/9912001.
[16] J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. **83** 4245 (1999) [hep-ph/9906523].
[17] E. E. Flanagan, S. H. H. Tye, I. Wasserman, hep-ph/9910498.
[18] P. Binétruy, C. Deffayet, U. Ellwanger and D. Langlois, hep-th/9910219.
[19] S. Mukohyama, Phys. Lett. **B473**, 241 (2000) [hep-th/9911165].
[20] D. N. Vollick, hep-th/9911181.
[21] P. Kraus, JHEP **9912**, 011 (1999) [hep-th/9910149].
[22] D. Ida, gr-qc/9912002.
[23] S. Mukohyama, T. Shiromizu and K. Maeda, hep-th/9912287, to appear in Phys. Rev. D.
[24] D. Birmingham, Class. Quant. Grav. **16**, 1197(1999).
[25] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[26] S. Wands, et. al., astro-ph/0003278.
[27] J. Garriga and M. Sasaki, hep-th/9912118.
[28] K. Koyama and J. Soda, gr-qc/0001033.
[29] S. W. Hawking, T. Hertog, and H. S. Reall, hep-th/0003052.

[30] For the proof of the uniqueness, see for example, S. Weinberg, *Gravitation and cosmology* (John Willey & Sons, Inc., 1972).
[31] For properties of these spacetimes, see for example, S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, (Cambridge University Press, 1973).
[32] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[33] H. Kodama, H. Ishihara and Y. Fujiwara, Phys. Rev. **D50**, 7292 (1994).
[34] A. Ishibashi and H. Ishihara, Phys. Rev. **D56**, 3446 (1997).
[35] W. Isarel, Nuovo Cim. **44B**, 1 (1966).
[36] K. Uzawa, Y. Morisawa and S. Mukohyama, gr-qc/9912108.
[37] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[38] G. Hayward and J. Louko, Phys. Rev. **D42**, 4032 (1990).
[39] N. Y. Vilenkin and Y. A. Smorodinskiĭ, Sov. Phys. JETP, **19**, 1209 (1964).
[40] M. A. Rubin and C. R. Ordóñez, J. Math. Phys. **25**, 2888 (1984); **26**, 65 (1985).

TABLE I. Two sets of values of (α, β, γ)

Φ	α	β	γ	α	β	γ
$\Phi_{(S)}$	$D-4$	1	0	$-(D-6)$	$D-4$	$2(D-5)$
$\Phi_{(V)}$	$D-2$	0	$-(D-3)$	$-(D-4)$	$D-3$	$D-3$
$F_{(T)}$	D	$-(D-3)$	$-2(D-2)$	$-(D-2)$	2	2

TABLE II. Δ

Φ	k	Δ
$\Phi_{(S)}$	$k^2[k^2-(D-2)K] \neq 0$	0
	$k^2 = 0$ and $K \neq 0$	Constant $\times\ r$
	$k^2 = (D-2)K$	Solution $\Delta_{(S)}$ of (47)
$\Phi_{(V)}$	$k^2 \neq (D-3)K$	0
	$k^2 = (D-3)K$	Constant
$F_{(T)}$	${}^{\forall}k$	0